בס"ד

Rambam and the Seasons

An astronomical look at the three methods given by *Rambam* for calculation of equinox or solstice moments

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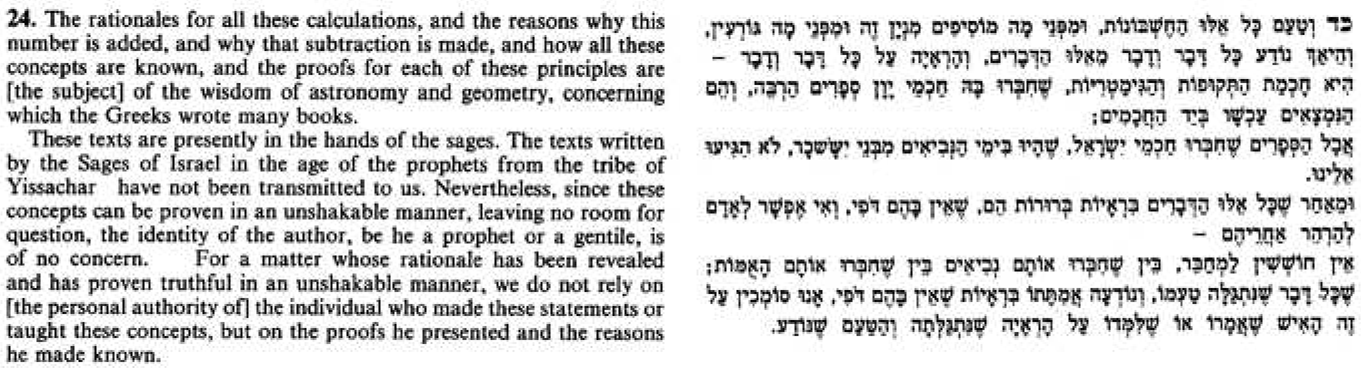
Menu of Topics:

Introduction: *Hilchot Kiddush haChodesh*

Rabbi Moshe ben Maimon ("*Rambam*"), also commonly known by his Greek name, (Moses) Maimonides, wrote a book entitled *Hilchot Kiddush haChodesh* (title translated as "Sanctification of the New Month" or alternatively as "Sanctification of the New Moon") around the Julian year 1178 or Hebrew year 4938, which is the eighth treatise in *Zmanim* (The Book of Seasons), the third book of his *Mishneh Torah* collection (code of Jewish Law).

As it will be necessary for me to present the results of astronomical calculations from non-Jewish sources, allow me to first quote what *Rambam* wrote about relying upon such sources, from the end of chapter 17 (click here or on the image to open a higher-resolution PDF version 73KB):



24. The rationales for all these calculations, and the reasons why this number is added, and why that subtraction is made, and how all these concepts are known, and the proofs for each of these principles are [the subject] of the wisdom of astronomy and geometry, concerning which the Greeks wrote many books.

These texts are presently in the hands of the sages. The texts written by the Sages of Israel in the age of the prophets from the tribe of Yissachar have not been transmitted to us. Nevertheless, since these concepts can be proven in an unshakable manner, leaving no room for question, the identity of the author, be he a prophet or a gentile, is of no concern. For a matter whose rationale has been revealed and has proven truthful in an unshakable manner, we do not rely on [the personal authority of] the individual who made these statements or taught these concepts, but on the proofs he presented and the reasons he made known.

כד וְטַעַם כָּל אֵלּוּ הַחֶשְׁבּוֹנוֹת, וּמִפְּנֵי מָה מוֹסִיפִים מִנְיָן זֶה וּמִפְּנֵי מָה גּוֹרְעִין,
וְהֵיאַךְ נוֹדַע כָּל דָּבָר וְדָבָר מֵאֵלּוּ הַדְּבָרִים, וְהָרְאָיָה עַל כָּל דָּבָר וְדָבָר –
הִיא חָכְמַת הַתְּקוּפוֹת וְהַגִּימַטְרִיּוֹת, שֶׁחִבְּרוּ בָּהּ חַכְמֵי יָוָן סְפָרִים הַרְבֵּה, וְהֵם
הַנִּמְצָאִים עַכְשָׁו בְּיַד הַחֲכָמִים;
אֲבָל הַסְּפָרִים שֶׁחִבְּרוּ חַכְמֵי יִשְׂרָאֵל, שֶׁהָיוּ בִּימֵי הַנְּבִיאִים מִבְּנֵי יִשָּׂשכָר, לֹא הִגִּיעוּ
אֵלֵינוּ.
וּמֵאַחַר שֶׁכָּל אֵלּוּ הַדְּבָרִים בִּרְאָיוֹת בְּרוּרוֹת הֵם, שֶׁאֵין בָּהֶם דֹּפִי, וְאִי אֶפְשָׁר לְאָדָם
לְהַרְהֵר אַחֲרֵיהֶם –
אֵין חוֹשְׁשִׁין לַמְחַבֵּר, בֵּין שֶׁחִבְּרוּ אוֹתָם נְבִיאִים בֵּין שֶׁחִבְּרוּ אוֹתָם הָאֻמּוֹת;
שֶׁכָּל דָּבָר שֶׁנִּתְגַּלָּה טַעְמוֹ, וְנוֹדְעָה אֲמִתָּתוֹ בִּרְאָיוֹת שֶׁאֵין בָּהֶם דֹּפִי, אָנוּ סוֹמְכִין עַל
זֶה הָאִישׁ שֶׁאֲמָרוֹ אוֹ שֶׁלִּמְּדוֹ עַל הָרְאָיָה שֶׁנִּתְגַּלְּתָה וְהַטַּעַם שֶׁנּוֹדַע.

In ***Hilchot Kiddush haChodesh***, *Rambam* gave **three methods for computing the moment of an equinox or solstice**. Two of the methods are well known, and are based on the assumption that the seasons are equally spaced.

Astronomically the actual intervals between the equinoxes and solstices are not equal, and they vary over the ages — please see my web page "**The Lengths of the Seasons**".

On this web page, when I refer to the length of the solar year, the intended meaning, since the context is with regard to the Hebrew Calendar, is the average time in days and fraction of a day from one Spring Equinox until the next Spring Equinox, in Israel. In today's politically correct, hemisphere neutral terminology, referring to that equinox as the Northward Equinox, this could be described as the Mean Northward Equinoctial year.

Ritual Importance of Exact Calculations

Exact calculations are essential for calendar purposes, especially where ritual is involved, to avoid disputes. That is why *Rambam* described how to do *molad*, equinox, solstice, and Hebrew calendar calculations using only whole numbers and remainders. Likewise, that is why this web page quotes exact fractions in preference to decimal numbers.

The Directions of Sunrise and Sunset: *Talmud Bavli* tractate *Eruvin* page 56a

The *Talmud Bavli* tractate *Eruvin* page 56a, in the context of surveying a city to determine its orientation with respect to the 4 cardinal directions prior to setting up an *eruv*, very briefly mentions that on *Tekufat Nisan* (northward or spring equinox, near the beginning of the Hebrew month of *Nisan*) and on *Tekufat Tishrei* (southward or autumn equinox, in the Hebrew month of *Tishrei*, near *Yom Kippur*), Sun rises at the middle of the range of sunrise points during the year and sets at the middle of the range of sunset points. This definition is astronomically quite valid for all inhabited locales on planet Earth, within plus or minus a day or so of the actual equinox, but it is an observational method that doesn't yield a way to compute the moment of any equinox. This method is also rather inconvenient because it requires a very long period of observation (at least from one solstice to the opposite solstice), which perhaps explains why it was apparently never used to support the *Sanhedrin* Calendar Committee's decision to declare a leap year. The insertion of the leap month had to be announced well in advance of the spring equinox, so that pilgrims would know when to begin their journey to arrive in Jerusalem for Passover.

The *Talmud* didn't specify whether the observer is to note the direction to the upper limb, center, or bottom limb of the solar disk. Atmospheric refraction near both horizons severely limits the accuracy of this observational method, especially on a cold morning with high humidity pressure inversion over the eastern horizon, which can make Sun appear to rise several minutes prematurely (4 minutes earlier per degree of refraction, and as much as +2° of refraction is possible). Observation from an elevated point also introduces errors, making the apparent sunrise earlier and the apparent sunset later. To minimize the error due to atmospheric refraction, the recorded direction should be that of the center of the solar disk at the moment that the bottom limb of Sun is just touching the horizon, and to minimize the error due to elevation, the observation should be made as close to sea level as possible. Refraction variability is less when observing sunrise and sunset at sea rather than on land, because the atmospheric conditions are less variable, but it would be difficult to permanently mark the observed directions at sea, unless observing from a small island.

A comparable method, not mentioned in the *Talmud*, but requiring only a brief observation period (a few days near the date of an equinox), is to draw a line from the direction of sunrise to the observer, and another line from the direction of sunset to the observer. That day was an equinox day if those two lines connect as one straight line, or if they come closer to doing so than on the prior or next day.

More precisely, the Sun would rise exactly due East only if Earth had no atmosphere, Sun were a point of light instead of a disk, and the equinox moment coincided with the moment of sunrise for the observer's sea-level locale, likewise it would set exactly due West only if the equinox moment coincided with the moment of sunset at the observer's locale. Obviously Sun can never do both on the same day for any given locale, it is rare for the equinox moment to coincide with either sunrise or sunset for any given locale, although for every equinox there is always exactly one meridian of longitude somewhere on Earth that sees sunrise at the moment of the equinox, and exactly one other meridian elsewhere (one daytime span to the East) that sees sunset at the moment of the equinox. In modern astronomical terminology the date of an equinox could be described as the day when the sunrise **azimuth** is 90° east of north or when the sunset azimuth is 90° west of north.

If the sunrise and sunset directions are observed at Earth's equator, then the errors due to refraction and elevation will be negligible, because at the equator Sun rises and sets nearly perpendicularly to the horizon. Also, at the equator when it is local apparent noon on the day of an equinox, sunlight will shine straight down any vertical shaft, and no object will cast a shadow, especially when the equinox moment is close to noon. Requiring that observations be made at the equator, however, is not very convenient for inhabitants of the Holy Land!

Click here to see charts of the **variation of the direction of sunrise and sunset with latitude at the equinoxes and solstices** 42KB.

The **direction** or **azimuth** of sunrise or sunset must be distinguished from the **angle** made by the path of the rising or setting Sun with respect to the horizon, known as the solar **parallactic angle**. Sun rises at the true east direction and sets at the true west direction on the day of an equinox, provided that the solar parallactic angle is 90°, which applies only at the equator.

At non-equatorial latitudes, due to atmospheric refraction near the horizon and the solar semi-diameter, especially at elevations above sea level, the upper solar limb is seen to rise before it reaches the direction observed at the equator (Jean Meeus, Mathematical Astronomy Morsels V, chapter 63, "Where does the Sun rise at the equinoxes?", pages 343-4), and for the same reasons the upper solar limb remains visible as Sun sets until after it passes the direction observed at the equator. The range of variation of sunrise and sunset directions are least at the equator, where they are equal to the Earth equatorial obliquity (axial tilt relative to the plane of Earth's orbit around Sun). At higher latitudes the variations are progressively greater, as shown in the following charts:

Click here to see charts of the **variation of the direction of sunrise and sunset with latitude at the equinoxes and solstices** 42KB.

At sunrise or sunset only, the solar parallactic angle is given by:

solar parallactic angle at sunrise or sunset = *arccosine*[*sine*(geographic latitude) / *cosine*(solar declination)]

where all angles are in degrees (Jean Meeus, Astronomical Algorithms, second edition, chapter 14, "The Parallactic Angle", on page 99). The parallactic angle can only be 90° on days when the solar declination equals the observer's geographic latitude, which occurs twice per year for locales between the Tropic of Cancer and Tropic of Capricorn, but *never* occurs at latitudes further north or south of the tropics. Locales that are on the Tropic of Cancer see a 90° parallactic angle only when sunset or sunrise occurs near the moment of the north solstice, and locales that are on the Tropic of Capricorn see a 90° parallactic angle only when sunset or sunrise occurs near the moment of the south solstice.

Daytime and Night Time Lengths are *NOT* Equal at Astronomical Equinoxes

The English word "equinox" comes from the Latin for "equal night", because the duration of daytime and night time are approximately equal on the date of the equinox. At both equinoxes the daytime length is actually slightly longer than the length of night, due to atmospheric refraction making Sun appear higher at sunrise and sunset, and due to the approximately $^1/_2$° diameter of the solar disk. Nevertheless, a properly set up sundial will show sunrise at 6 am and sunset at 6 pm on equinox days at equatorial latitudes, which explains why prior to the invention of mechanical clocks the length of day and night were believed to be equal on those days.

At the north solstice the daytime length is maximal in the northern hemisphere, and minimal in the southern hemisphere, and the converse applies to the south solstice.

Saying that the day and night are equal is the same as saying that the daytime temporal hour (*sha'ah zmanit*) is exactly 60 minutes, when calculated by the method of the *Vilna Gaon* as (sunset – sunrise) / 12.

Due to atmospheric refraction, when near either horizon Sun always appears to be at a higher apparent altitude than its true geocentric altitude, so daytime at the astronomical equinox is always longer than night time (measured on a mechanical, electric, electronic, or atomic clock). For example, at Jerusalem on the autumnal equinox in the Hebrew year 5765 the daytime was 12 hours and 17 minutes, whereas the night time was 11 hours and 42 minutes (this doesn't add up to exactly 24 hours because the seconds were rounded as they are insignificant and because it is already changing for the next day). That day was 35 minutes longer than the following night. For Jerusalem the day that actually has equal day and night in the autumn is 8 days later than the autumnal equinox, and in the spring it is 8 days earlier than the vernal equinox.

The number of days that separate the equinoxes from the day of equal day and night varies with latitude, with the least separation at higher latitudes where the length of the day changes more rapidly in the vicinity of the equinoxes. For example, in Toronto the equal day and night occurs 4 to 5 days after the autumnal equinox and about 5 days before the vernal equinox.

Click here to see charts of the **variation of daytime length with latitude at the equinoxes and solstices** 42KB.

Click here to see charts of the **variation of daytime length with latitude throughout the year** 133KB.This chart shows that in both the north and south hemisphere the dates of equal length daytime and nightime (where the daytime lengths cross the 12-hour line) are several days prior to the spring equinox and several days after the autumn equinox. The gaps separating the 12-hour days from the equinoxes are unequal and differ between hemispheres, because of differences in Earth's orbital velocity, and are shorter at higher latitudes (further away from the equator), because the length of day changes more rapidly near the equinoxes at latitudes further away from the equator, as shown in the chart.

Modern Astronomical Calculations

To find astronomically accurate equinoxes and solstices for the charts below, I used **numerical integration**, which is arguably the "gold standard" for celestial mechanics, and which is easy to do using **SOLEX**, a "postcard-ware" computer program written by Professor Aldo Vitagliano of the Chemistry Department at the University of Naples, Italy. For more information about SOLEX, please see my "Length of the Seasons" web page at <http://www.sym454.org/seasons/>.

The SOLEX integration was carried out in terms of Terrestrial Time (usually abbreviated TT but indicated as TDT within SOLEX), with Delta T switched off and the geographic locale set to the Equator at the Prime Meridian, over the range of dates corresponding to the first 10000 years of the Hebrew Calendar.

After integration, I calculated the required equinox and solstice moments by linear interpolation using geocentric ecliptic angular solar longitudes (relative to equinox of date) as output by SOLEX at the steps before and after each equinox and solstice. (If using SOLEX 9.1 or later, interpolation is no longer necessary for finding equinox or solstice moments by declination.)

To convert the interpolated SOLEX equinox and solstice moments from Terestrial Time (TT) to Universal Time (UT) one subtracts the corresponding Delta T (ΔT) value, for which I used the updated polynomial approximations published at the end of January 2007 by Jean Meeus and Fred Espenak at the NASA Eclipses web site <http://eclipse.gsfc.nasa.gov/SEhelp/deltatpoly2004.html>. To avoid monthly granularity (stepwise increments) in the Delta T approximation, however, I used the following expression when calculating *y* (the fractional year number):

$y = 2000 + (TTmoment - \text{J2000.0}) / MARY$

where *TTmoment* is the Terrestrial Time moment and J2000.0 is the moment corresponding to January 1, 2000 AD at 12:00h TT, both in terms of the number of days and fraction of a day elapsed relative to a specified ordinal day numbering epoch, and *MARY* (Mean Atomic Revolution Year) = $365+{}^{31}/_{128}$ atomic days, as explained on "The Lengths of the Seasons" at <http://www.sym454.org/seasons/>. Over the entire range from 500 BC to 2050 AD, this modification never causes more than ${}^{4}/_{5}$ second of difference compared to the unmodified arithmetic of the NASA algorithm.

Note that $\Delta T = TT - UT$, $UT = TT - \Delta T$, and (approximately, but adequate for the next 10000 years) $TT = UT + \Delta T$.

Finally, adding 2 hours and 21 minutes converts the Universal Time moments to **Jerusalem mean solar time**.

Method #1: *Tekufat Shmuel* (תקופת שמואל)

In chapter 9 *Rambam* discussed *Tekufat Shmuel* or Samuel (תקופת שמואל), attributed (by others) to *Amora* Shmuel of Nehardea ("Shmuel the Astronomer"), who assumed that the four *Tekufot* (seasons) are equal in length, each of $91+{}^{5}/_{16}$ days = 91 days $7+{}^{1}/_{2}$ hours, based on ${}^{1}/_{4}$ of the length of an assumed solar year of 365 days + 6 hours = $365+{}^{1}/_{4}$ days, which is the same as the mean length of the Julian Calendar year. (This year length was, extrapolated backwards, actually the length of the Northward Equinoctial year at around 100,000 BC when the Earth was rotating slightly faster. Currently the Mean Northward Equinoctial year is around 365 day 5 hours 49 minutes 0 seconds, which is 12 seconds shorter than the mean Gregorian Calendar year length of $365+{}^{97}/_{400}$ = 365 days 49 minutes 12 seconds, and continuing to get shorter as Earth's rotation slows due to the tides. The mean year of *Tekufat Shmuel* or the Julian Calendar is currently 11 minutes too long.)

This method is also partially discussed in *Talmud Bavli* tractate *Eruvin* near the bottom of page 56a, quoting Shmuel, but the details given there are insufficient to reproduce the calculation.

The moment of any equinox or solstice according to this method is calculated by starting with the traditional moment of the spring equinox of Creation, taken in this case as the sunset 7 days, 9 hours and 642 parts prior to *molad* of Nisan of year 1 at *rata die* –1373257.25 = Julian day number 348167.25, then adding the number of seasons of length $91+{}^{5}/_{16}$ days that have elapsed since that epoch.

The traditional derivation of the epoch of *Tekufat Shmuel* relative to the *molad* was as follows (*Tosefot* on *Talmud Bavli* tractate *Rosh HaShanah* page 8a at *L'Tekufot*):

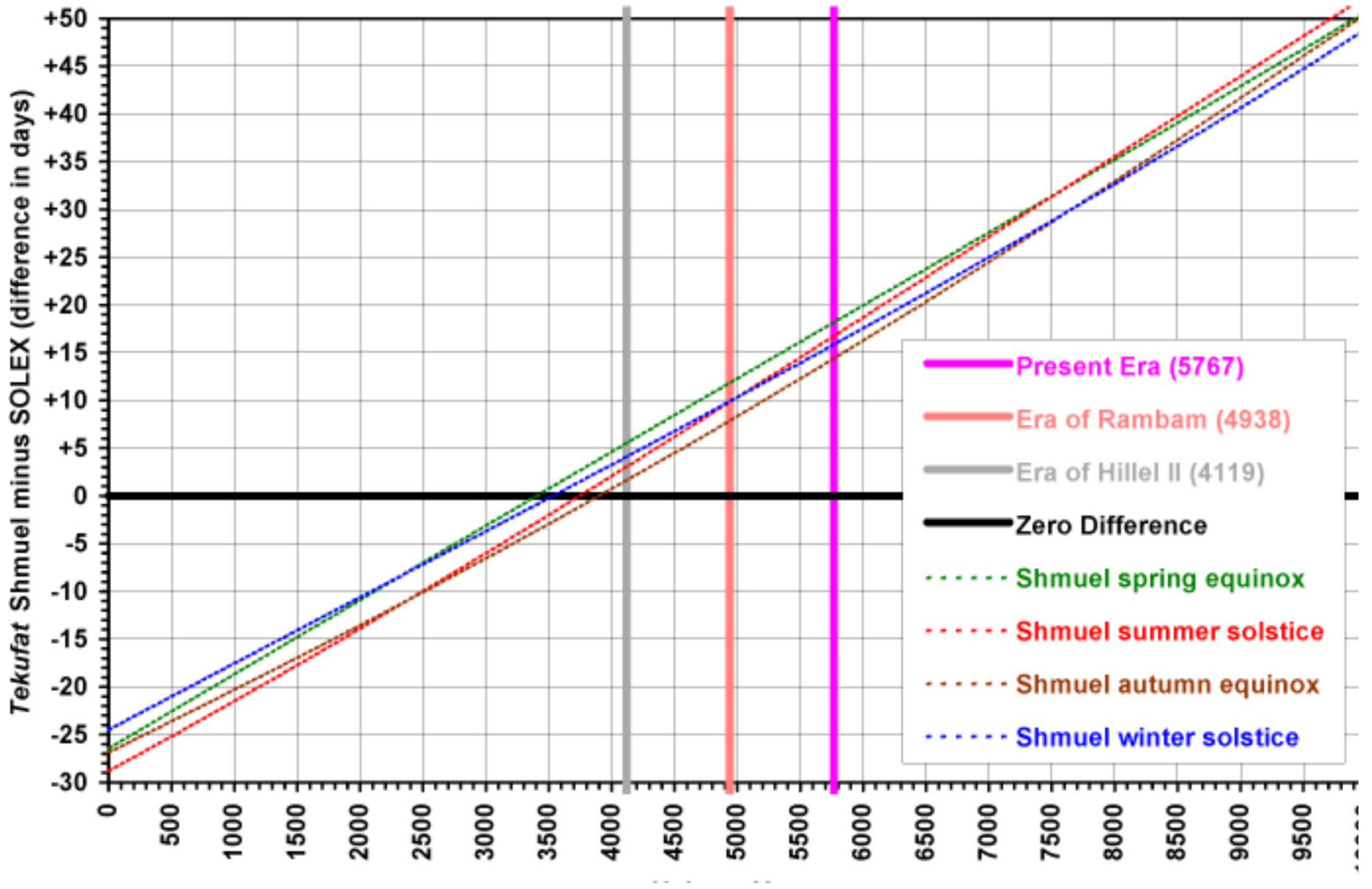
- *Adam haRishon* was considered to have seen the first lunar crescent at the start of the 9th hour of the daytime on the 6th day of Creation (20 hours from the sunset that started that date), when *haShem* commanded him not to eat from the Tree of Knowledge. The days of Creation are traditionally considered to have been the final days of Hebrew year 1, so this observation sanctified the month of *Tishrei* of year 2.
- Traditionally, assuming that the lunar conjunction was 6 hours earlier, the moment of the *molad* of the month of *Tishrei* of Hebrew year 2 was at the start of the 3rd hour of the daytime on Friday (14 hours from the sunset that started that date).
- The *molad* of *Nisan* of Hebrew year 1 was considered to have occurred 6 lunar months earlier, where

each lunar month equals the *molad* interval in duration (29 days 12 hours 44 minutes 1 part).

- The seasons were reckoned starting from the spring equinox in *Nisan* (*Tekufat Nisan*), assuming that Sun was created at the sunset that began the 4th day (Hebrew weekday = Wednesday).
- The next fall equinox (*Tekufat Tishrei*), prior to *Tishrei* of Hebrew year 2, was two seasons later. According to *Tekufat Shmuel*, each season has 13 weeks, which can be ignored, plus a remainder of $^{5}/_{16}$ day = $7+^{1}/_{2}$ hours, so the fall equinox must have been $2 \times 7+^{1}/_{2} = 15$ hours after sunset (3 hours after sunrise, or the start of the 4th hour of the day) on the last Wednesday in *Elul*, which was 1 day and 23 hours prior to the *molad* of *Tishrei*.
- The difference between 6 solar months and 6 lunar months = $(365+^{1}/_{4}) / 2 - (6 \times$ the traditional *molad* interval$)$ = 5 days 10 hours 642 parts.
- Therefore the first *Tekufat Nisan* (in Hebrew year 1) was 1 day and 23 hours + 5 days 10 hours 642 parts = 7 days 9 hours and 642 parts **before** the *molad* of *Nisan*.

The following chart compares the SOLEX equinox and solstice moments with *Tekufat Shmuel* (dashed lines) from the traditional year of Creation for the first 10 millennia of the Hebrew Calendar. Although the seasons of *Tekufat Shmuel* pass at fixed intervals, it is impossible to quote a constant error rate taking the actual astronomy as a reference, beause the astronomical changes are complex. Clicking here or on the chart will open a larger, higher resolution version, in Adobe Acrobat PDF format (42KB) that compares *Tekufat Shmuel* with the other methods discussed on this page.

Tekufat Shmuel vs. SOLEX Equinox and Solstice Moments (Jerusalem mean solar time)



In comparison with the astronomical equinoxes and solstices, the calculation of Shmuel yields equinox and solstice moments that currently drift about one day later for each elapsed 130.9 years (the reciprocal of 11 minutes expressed as a fraction of a day). Today the Shmuel estimate for the autumn equinox is more than $14+{}^{5}/_{11}$ days later than the mean astronomical Southward Equinox. The Shmuel estimate of the spring equinox moment is appreciably worse, currently almost $18+{}^{1}/_{4}$ days late with respect to the mean astronomical Northward Equinox.

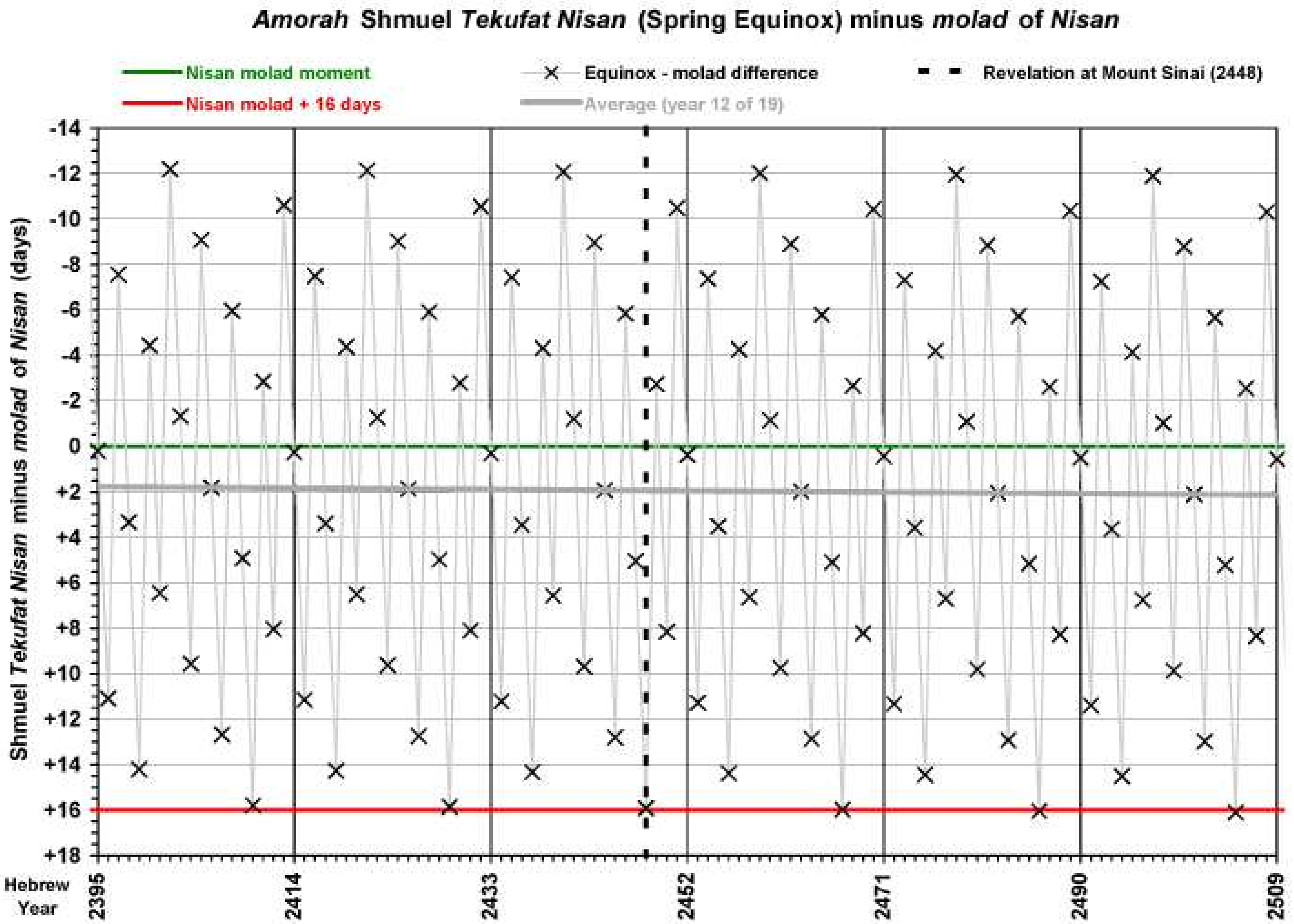
The *Tekufat Shmuel* calculation was probably established near the era when the Shmuel minus SOLEX spring equinox difference was closest to zero. That was around Hebrew year 3400, which was well before the era of Shmuel, near the era when the second *Beit HaMikdash* (second temple in Jerusalem) was established. An authority that taught about this method prior to Shmuel was Rebbe Eliezer ben Hurcanus, traditional author of *Pirkei D'Rebbi Eliezer*, which describes this method in considerable detail (chapters 6 and 7). Presumably this method is known as *Tekufat Shmuel* instead of *Tekufat Eliezer* because the latter's teachings weren't assembled and published until several centuries after Shmuel, and because the *Talmud* quoted Shmuel in regard to this method.

The back-calculated Traditional Equinox of Creation according to *Tekufat Shmuel* was astronomically about 26 days too early, the Shmuel summer solstice that year was about 29 days too early, and the timing of that solstice has the worst drift rate over the span of years shown. The NASA Delta T for that era was 1 day 3 hours and almost 40 minutes, so if the uncertainty is better than 25% then my astronomical start of season moments ought to be accurate to within less than 7 hours.

Despite its obviously severe drift with respect to the seasons, *Tekufat Shmuel* is still ritually used today to calculate the autumn equinox for the purpose of *Sh'ela* (Request for Rain). Others have suggested that it is used because its calculation is "simpler" than *Tekufat Adda* (תקופת אדא), but the "extra complexity" of the latter method is surely negligible, because the only differences are 7 days at the epoch and a season length difference of 1 minute and $8+{}^{3}/_{4}$ seconds = $20+{}^{5}/_{8}$ parts. I think it more likely that in the era when the decision was made to use *Tekufat Shmuel* for the purpose of *Sh'ela* it was the only method available and at that time it yielded autumnal equinox moments that were very close to the contemporary astronomical observations, shown in the chart as shortly after Hebrew year 3900, which was the era when Shmuel lived (3925 to 4017).

The year according to *Shmuel* is 365 days and 6 hours = $365+{}^{1}/_{4}$ days in length, whereas the mean year of the traditional Hebrew calendar = *molad* interval × 235 months / 19 years = 365 days 5 hours 55 minutes and ($25+{}^{25}/_{57}$ seconds <u>or</u> $7+{}^{12}/_{19}$ *chalakim*) ≈ 365.2468222 days. In other words, the year of *Tekufat Shmuel* is exactly 4 minutes and $34+{}^{32}/_{57}$ seconds = ${}^{313}/_{98496}$ of a day longer than the Hebrew calendar mean year, so its average drift rate toward dates that average progressively later in the Hebrew calendar year is the inverse of that fraction = ${}^{98496}/_{313} = 314+{}^{214}/_{313} \approx 314.6837$ Hebrew years per day of drift. *Rambam* gave the difference per 19-year cycle as 1 hour and 485 *chalakim* (chapter 6 #10 and chapter 9 #4) = ${}^{1}/_{24}+{}^{485}/_{25920} = {}^{313}/_{5184}$ of a day per cycle. Dividing by 19 yields ${}^{313}/_{98496}$ of a day, exactly in agreement with the above.

If *Tekufat Shmuel* is so obviously inaccurate then why does it have a place in our tradition? *Rambam* wrote that it was because its calculations are simpler than the more accurate method, *Tekufat Adda*, discussed below, but I don't think that was the real reason, because the calculations are nearly identical, only differing in the assumed epoch (one week difference) and the assumed mean year. I think that the real reason is disclosed by the following chart, which shows the back-calculated difference between *Tekufat Shmuel* and the *molad* of *Nisan* in the era of the Revelation at Mount Sinai (click here or on the chart to open a higher-resolution PDF version, 20 KB):



Amorah Shmuel *Tekufat Nisan* (Spring Equinox) minus *molad* of *Nisan*

Year 16 of every 19-year cycle always has the latest equinox moments. The traditional year of the Revelation at Mount Sinai, Hebrew year 2448, was the 16th year of 19, and its *Tekufat Nisan* according to Shmuel was less than 16 days after the *molad* of *Nisan*! Traditionally, that was the year of the biblical exodus from Egypt. The ancient Egyptians based their calendar year on the heliacal rising of Sirius, starting a new year on the first morning when Sirius was visible rising ahead of Sun, some number of days after the north solstice. The Nile used to flood their land soon afterward. The Sirius heliacal rising mean year length is to good accuracy $365+{}^{1}/_{4}$ mean solar days, which equals the *Tekufat Shmuel* mean year!

Apparently it wasn't realized that the mean northward equinoctial year (from spring equinox to spring equinox) is significantly shorter than the mean Sirius heliacal rising year.

One should not falsely conclude from the above chart that *Tekufat Shmuel* was accurate in the era of the exodus, because in fact at that time, compared to the mean astronomical northward equinox, it was running more than a week too early. Nevertheless the relationship between the *molad* of *Nisan* and *Tekufat Shmuel* in that era can't be coincidental.

Looking for further astronomical correlations, it is also intriguing that a **total solar eclipse** crossed the populated area of northern Egypt shortly after midday on Saturday, May 14, 1338 BC (Julian date, no year zero) = 29 *Iyar* 2423 (25 years before the traditional date of the Revelation at Sinai), as shown in this map from the NASA Eclipses web site <http://eclipse.gsfc.nasa.gov/SEatlas/SEatlas-2/SEatlas-1339.GIF>, with further details shown in this schematic diagram <http://eclipse.gsfc.nasa.gov/5MCSEmap/-1399--1300

/-1337-05-14.gif>. Note that NASA web pages show the year number as 1337 BC or -1337 because their calculations include a year zero. That eclipse was the only total solar eclipse that passed through Egypt near that era. That event occurred about $41+{}^{2}/_{3}$ days after the northward equinox, so by astronomical criteria that was a Hebrew leap year so it was *Rosh Chodesh Iyar* not *Sivan*. Moon was near perigee (the point in its orbit when it is closest to Earth) so its disk diameter was larger than Sun, and Earth was near aphelion (the point in its orbit when it is furthest from Sun, hence the apparent solar disk diameter was near its minimum, for example see this photo at NASA: <http://antwrp.gsfc.nasa.gov/apod/ap070709.html>), making it a very dark and near-maximal-length eclipse. The duration of the total eclipse phase was just under 7 minutes, however, not quite a "plague of utter darkness lasting 3 days" — could it be that legend grossly exaggerated that event?

It may also be significant that in Hebrew year 2204, which was 244 years earlier than the traditional year of Revelation at Mount Sinai, *Tekufat Shmuel* was essentially equal to *Tekufat Adda*, as will be discussed below.

Relative to the *molad*, in each 19-year cycle the moments of *Tekufat Shmuel* are exactly ${}^{313}/_{5184}$ of a day later than the previous cycle. The annual drift rate is ${}^{1}/_{19}$ of that = ${}^{313}/_{(5184 \times 19)}$ = ${}^{313}/_{98496}$ of a day, so again the number of years for the mean *Tekufat Shmuel* to drift exactly one day later relative to the *molad* (or the Hebrew calendar) is the inverse of that fraction = ${}^{98496}/_{313} = 314+{}^{214}/_{313} \approx 314.6837$ Hebrew years per day of drift.

Many modern authorities assume, claim, or assert that the moments of *Tekufat Shmuel* refer to the meridian of Jerusalem, as if to say that they are in terms of the local mean time in Jerusalem. *Rambam* made no such claim, however, and I am aware of no primary traditional source for such a claim. The point is moot anyhow, given the grossly erroneous drift of *Tekufat Shmuel* with respect to the actual astronomical seasons, calculated for *any* meridian.

Method #2: *Tekufat Adda* (תקופת אדא)

In chapter 10, *Rambam* described as a "slightly more accurate calculation", attributed (by others) to Rav Adda bar Ahavah (רב אדא בר אהבה) and known as *Tekufat Adda* (תקופת אדא), based on a solar year length of 365 days + 5 hours + 997 parts + 48 *regaim* (a part is ${}^{1}/_{18}$ of a minute, a *rega* is ${}^{1}/_{76}$ of a part), which was derived from the length of an assumed solar year based on the *molad* interval × 235 lunar months per 19 solar years = 29 days 12 hours 44 minutes 1 part × ${}^{235}/_{19}$ = 365 days 5 hours 55 minutes and $25+{}^{25}/_{57}$ seconds ≈ 365.2468222 days. This year length is identical to the mean length of the Hebrew Calendar year, and is currently about 6 minutes and 25 seconds longer than the mean Northward Equinoctial year length (and corresponds, extrapolated backwards, to the Mean Northward Equinoctial year length of around 60,000 BC).

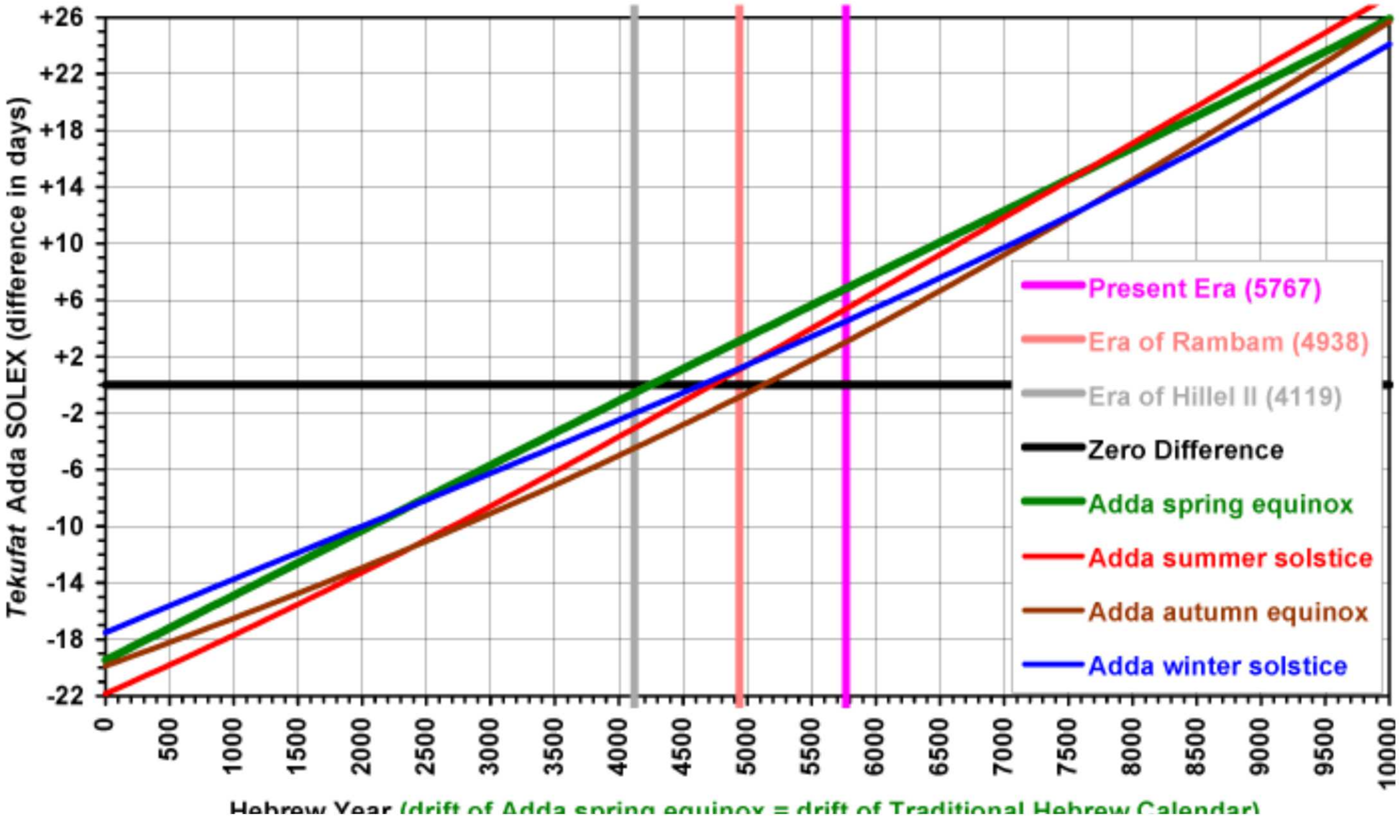
The moment of any equinox or solstice according to this method can be calculated by starting with the traditional moment of the Vernal Equinox of Creation, taken in this case as the sunset 9 hours and 642 parts prior to *molad* of *Nisan* of year 1 at *rata die* –1373250.25 = Julian day number 348174.25 (note that this epoch was exactly 7 days after epoch used for *Tekufat Shmuel*, above), then adding the number of equal-length seasons elapsed since the epoch, where the length of each season is exactly ${}^{1}/_{4}$ of the assumed solar year length = 91 days 7 hours 519 parts and 31 *regaim* = 91 days 7 hours 28 minutes and $51+{}^{41}/_{114}$ seconds per season. On average, this method has less than half the rate of seasonal drift compared to method #1 above.

The traditional derivation of the epoch of *Tekufat Adda* relative to the *molad* was as follows:

- *Adam haRishon* was considered to have seen the first lunar crescent at the start of the 9th hour of the daytime on the 6th day of Creation (20 hours from the sunset that started that date), when *haShem* commanded him not to eat from the Tree of Knowledge. The days of Creation are traditionally considered to have been the final days of Hebrew year 1, so this observation sanctified the month of *Tishrei* of year 2.
- Traditionally, assuming that the lunar conjunction was 6 hours earlier, the moment of the *molad* of the month of *Tishrei* of Hebrew year 2 was at the start of the 3rd hour of the daytime on Friday (14 hours from the sunset that started that date).
- The *molad* of *Nisan* of Hebrew year 1 was considered to have occurred 6 lunar months earlier, where each lunar month equals the *molad* interval in duration (29 days 12 hours 44 minutes 1 part), which was 9 hours and 642 parts after sunset on a Wednesday (4th day of the week).
- The seasons were reckoned starting from the spring equinox in *Nisan* (*Tekufat Nisan*), assuming that Sun was created at the sunset that began the **same** Wednesday. Therefore the epoch for *Tekufat Nisan* was 9 hours and 642 parts before the *molad* of *Nisan*.
- Thus the epoch of *Tekufat Adda* was exactly 7 days after the epoch of *Tekufat Shmuel*.

The following chart compares the SOLEX equinox and solstice moments with *Tekufat Adda* (thick solid lines) from the traditional year of Creation for the first 10 millennia of the Hebrew Calendar. Although the seasons of *Tekufat Adda* pass at fixed intervals, it is impossible to quote a constant error rate taking the actual mean astronomy as a reference, beause the astronomical changes are complex. Clicking here or on the chart will open a larger, higher resolution version, in Adobe Acrobat PDF format (42KB) that compares *Tekufat Adda* with the other methods discussed on this page.

Tekufat Adda vs. SOLEX Equinox and Solstice Moments (Jerusalem mean solar time)

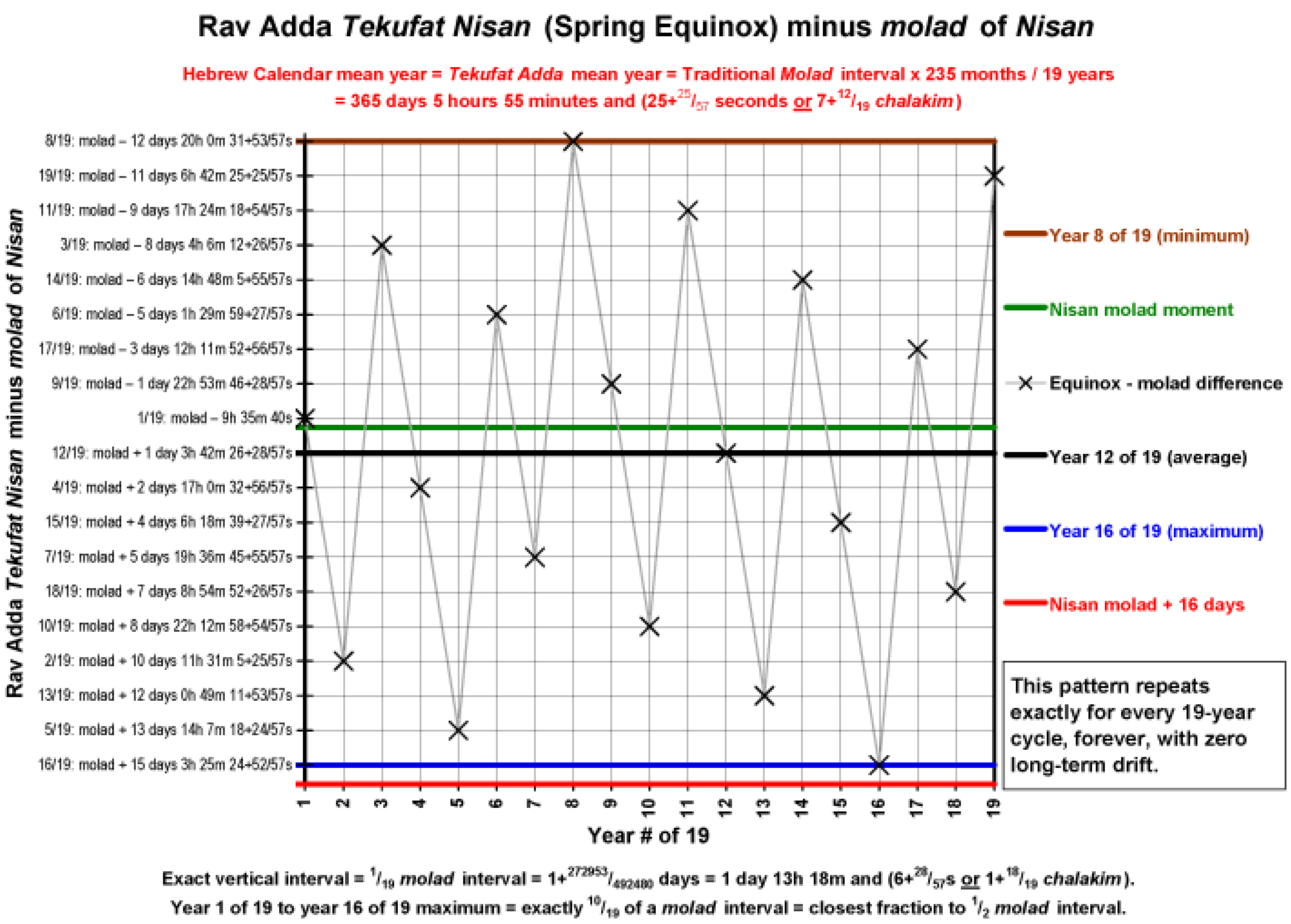


The *Tekufat Adda* calculation was probably established near the era when the Adda minus SOLEX spring equinox difference was closest to zero. That was near the era of Hillel ben Yehudah, when the traditional fixed arithmetic Hebrew Calendar was established. **Since the spring equinox moments according to *Tekufat Adda* were at that time very close to the actual astronomical equinox moments, it is reasonable to directly compare the alignment of the Traditional Hebrew Calendar of Hillel's era with respect to the astronomical Northward Equinox. It is impossible, however, to ever detect any drift between the Traditional Hebrew Calendar and any equinox or solstice moments of *Tekufat Adda*, because their respective mean year lengths are always identical (they drift at the same rate).**

Today the autumn equinox of *Tekufat Adda* is slightly more than $3+{}^{1}/_{8}$ days late, yet his spring equinox is almost 7 days late. These substantial differences in drift rate are due to the astronomical differences in the lengths of the equinoctial years and the astronomically unequal season lengths, as explained on my web page "**The Lengths of the Seasons**" at <http://www.sym454.org/seasons/>. Although the seasons of *Tekufat Adda* pass at constant intervals, it is not possible to quote a simple error rate taking the actual mean astronomy as a reference, beause the astronomical changes are complex.

The back-calculated Traditional Equinox of Creation according to *Tekufat Adda* was astronomically about $19+{}^{1}/_{2}$ days too early, and the Adda summer solstice that year was almost 22 days too early, and the timing of that solstice has the worst drift rate over the span of years shown. The NASA Delta T for that era was 1 day 3 hours and almost 40 minutes, so if the uncertainty is better than 25% then my astronomical start of season moments ought to be accurate to within less than 7 hours.

Tekufat Adda has a permanently fixed relationship **relative to the *molad*** of any month, following a pattern that exactly repeats every 19 years. For example, at *Nisan* of year 1 the tradition is that *Tekufat Nisan* of *Adda* was exactly 9 hours 35 minutes and 40 seconds before the *molad* moment (given above as 9 hours and 642 parts), and that relationship recurs in the first year of every 19-year cycle, as shown below (click here or on the image to open a higher resolution PDF version, 19 KB):



As is the case for *Tekufat Shmuel*, many modern authorities likewise assume, claim, or assert that the moments of *Tekufat Adda* refer to the meridian of Jerusalem, as if to say that they are in terms of the local mean time in Jerusalem. *Rambam* made no such claim, however, and I am aware of no primary traditional source for such a claim. The point is moot anyhow, given the grossly erroneous drift of *Tekufat Adda* with respect to the actual astronomical seasons, calculated for *any* meridian.

Many people mistakenly believe that *Tekufat Nisan* of *Rav Adda* is actually used for traditional Hebrew calendar arithmetic, but the truth is that the calendar employs a fixed 19-year leap cycle together with a fixed *molad* interval. It is nevertheless *possible* to employ *Tekufat Nisan* of *Rav Adda* instead of the fixed leap cycle, but that method can only reproduce traditional Hebrew calendar dates if the correct equinox cutoff time is employed. It may surprise you to learn that the necessary rule has to be "insert a leap month before *Adar* if otherwise *Tekufat Nisan* of *Rav Adda* will land later than one hour before noon on the 16th of *Nisan*". Any alternative cutoff moment will sometimes fail to reproduce traditional Hebrew calendar dates. Equivalently, the calendar could be regulated relative to the *molad* of *Nisan*: "insert a leap month before *Adar* if otherwise *Tekufat Nisan* of *Rav Adda* will land more than 16 days after the *molad* of *Nisan*".

Method #3: Solar Longitude

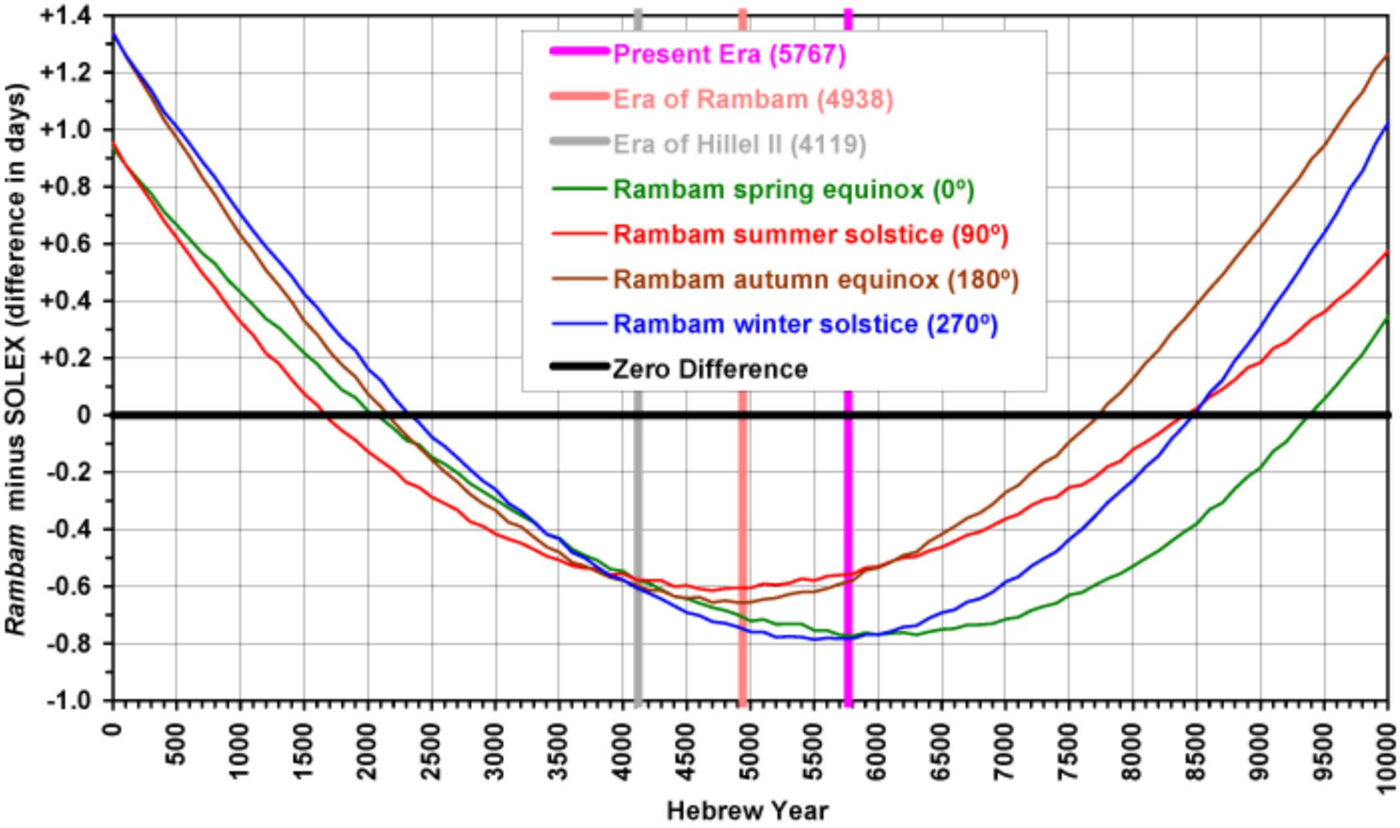
At the end of chapter 13 in point #11 *Rambam* gave, almost as a footnote or an afterthought, a third method which turns out to be quite accurate and may have been ignored or misunderstood by most of his readers. After describing his method for calculating the **true solar longitude** (using the longitude of the solar apogee, which today we call aphelion, to adjust the mean solar longitude), *Rambam* finished the chapter by explaining that **the moment of an equinox or solstice can be determined by searching for the moment which has the appropriate true solar longitude** (0°=spring equinox or *Tekufat Nisan*, 90°=summer solstice or *Tekufat Tammuz*, 180°=autumn equinox or *Tekufat Tishrei*, 270°=winter solstice or *Tekufat Tevet*).

At the end of chapter 11 *Rambam* specified that his own astronomical algorithms refer to Jerusalem and the region of Israel. Therefore it is valid to compare them to modern reckoning of Jerusalem mean local time.

Even a quick review of *Rambam*'s method makes it obvious why it was not adopted for ritual purposes, for it involves a prolonged series of calculations that must be repeated until the required moment is found. In our era of programmable computers, however, such calculations are easily executed in less than the "blink of an eye". I used Visual Basic 6 to implement *Rambam*'s true solar longitude algorithm, and then added a loop that iteratively searches for the moment when the longitude corresponds to a specified equinox or solstice, using a progressively converging binary-search algorithm similar to the *solar-longitude-after* function of Dershowitz & Reingold— for algorithm references please see my web page "**The Lengths of the Seasons**" at <http://www.sym454.org/seasons/>. Typically the execution loop repeats 20 times to converge on the moment of each equinox or solstice, so calculating the moments of all of the equinoxes and solstices for the first 10000 years of Hebrew calendar requires executing *Rambam*'s algorithm about 800,000 times, yet this takes only a few seconds on a modern personal computer system!

I compared *Rambam*'s true solar longitude algorithm results to the modern astronomical numerical integration calculations of SOLEX. The following chart compares the SOLEX equinox and solstice moments with those computed as described using *Rambam*'s solar longitude algorithm, from the traditional year of Creation for the first 10 millennia of the Hebrew Calendar. Clicking here or on the chart will open a larger, higher resolution version, in Adobe Acrobat PDF format (21KB), but unlike the charts used for the other methods discussed on this page, in this case the *Y*-axis is stretched to allow detailed examination of the much smaller differences:

The Seasons According to *Rambam*'s True Solar Longitude Algorithm compared to SOLEX Equinox and Solstice Moments (Jerusalem mean solar time)



Reasons for Small Errors in the Solar Longitude Method of *Rambam*

Over the Hebrew year range from 3000 to 6500 the *Rambam* equinox and solstice moments were/will be always earlier than the modern estimate, but the maximum difference is less than $^3/_4$ day for the South Solstice around year 5500, and the maximum difference of just over $^1/_2$ day for the Southward Equinox was actually during the era of *Rambam*. The minimum difference in that range of years was about $^1/_5$ day early for the South Solstice and Northward Equinox around the year 3000. **Therefore, if today this solar longitude method were used to calculate the date of the *Tekufot* (in particular *Tekufat Nisan*, the spring equinox), the calculated equinox and solstice moments would always land within one day of the correct moment** (for the next few thousand years). Several reasons for differences between the solar longitude results of *Rambam* and SOLEX are:

1. *Rambam* could not have known that the rotation of Earth is slowing down due to tidal forces, because that wasn't suspected until the Gregorian 19th century and not confirmed until the 20th century. Modern calculations include an approximate Delta T adjustment to account for the rotational slowing. The lack of any such adjustment is the main reason for the long-term parabolic curvature of the *Rambam* seasons comparison chart shown above.
2. In the first paragraph of chapter 12, *Rambam* adopted a mean solar motion rate of 98° 33' 53" in 100 days, which is exactly 59' 8" 19''' 48'''' per day = $^{59}/_{60} + {}^{8}/_{(60 \times 60)} + {}^{19}/_{(60 \times 60 \times 60)} + {}^{48}/_{(60 \times 60 \times 60 \times 60)}$ = exactly $^{354833}/_{360000}$ of a degree per day. Inverting the fraction yields $^{360000}/_{354833}$ days per degree. Multiplying that inverted fraction by 360° yields his implied mean solar year = $365+{}^{85955}/_{354833}$ days = exactly 365 days 5 hours 48 minutes $49+{}^{212143}/_{354833}$ seconds, or about 365.242240716055 days or 365 days 5h 48m 49.6s, which is a few seconds too long relative to the astronomical mean tropical year. [The denominator 354833 never reduces because it is a full reptend prime number, so all fractions of it represented as

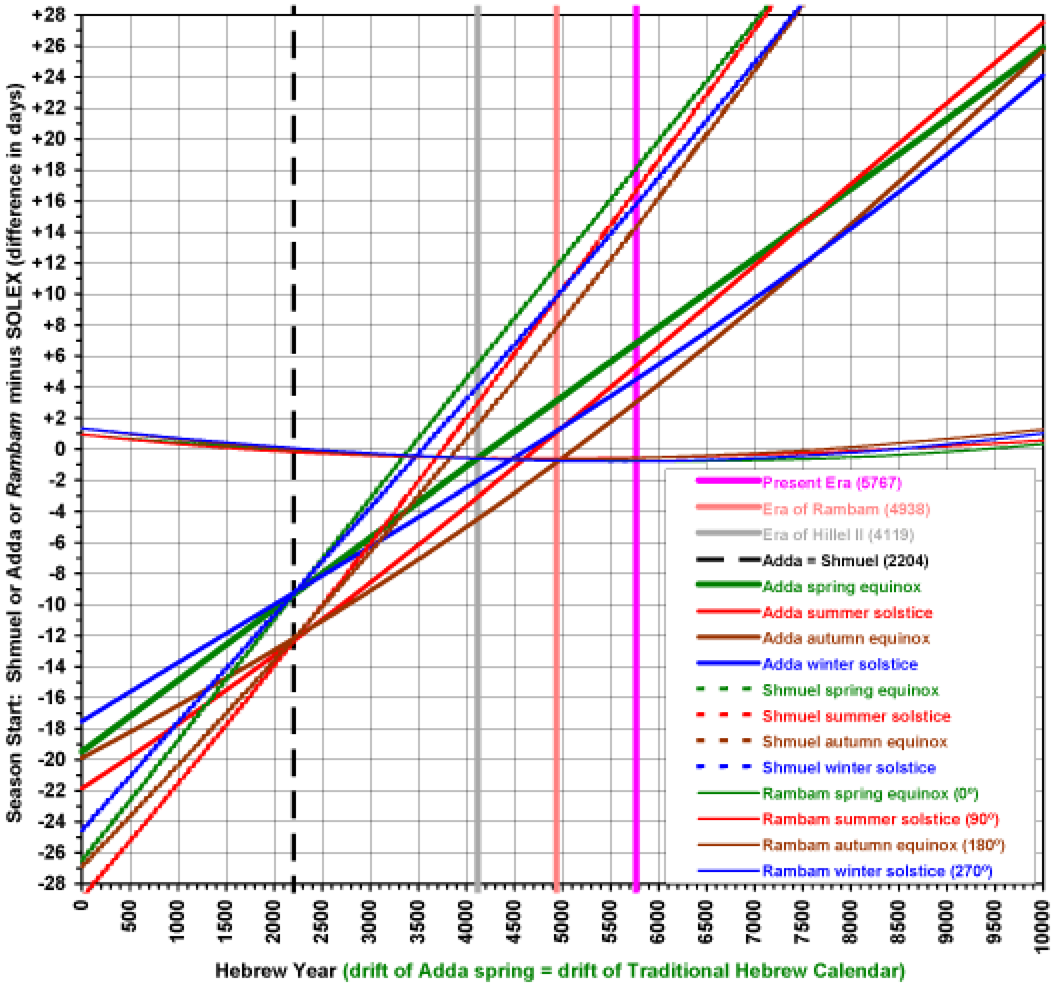
   decimal numbers have 354833-1=354832 repeating digits.]

3. *Rambam* assumed a constant rate of advance of the longitude of the solar "apogee" (so-called because he thought that Sun orbited Earth) amounting to about 1° per 70 years, which is somewhat slower than the actual mean rate of advance of aphelion of about 1° per 59 years (relative to the northward equinox of the date). This chart shows the longitude differences between the "apogee" of Maimonides and the "aphelion" of a modern astronomical calculation 16KB (=180° added to the result returned by Jean Meeus' perihelion polynomial from page 360 in "More Mathematical Astronomy Morsels", published 2002 by Willmann-Bell, Richmond, Virginia, USA). This difference makes a small contribution to the long-term paraboloidal curvature of the *Rambam* seasons comparison chart shown above. It would have been very difficult for classical astronomers to accurately measure the advance of aphelion, because of large observational year-to-year variations due to the mutual orbiting of Earth and Moon around their center of gravity (see page 399 in the Meeus book just cited). The chart shows that the best accuracy of the *Rambam* apogee longitude was around Hebrew year 3900 or the 2nd century AD of the Julian Calendar, corresponding to the era of Ptolemy of Alexandria, and indeed his apogee advance rate is the same as that published by Ptolemy in his book known as the *Almagest*.

Multi-Way Comparison

At the end of chapter 10, *Rambam* wrote that the "vernal equinox will take place approximately two days before the time determined by either of these calculations" (referring to the simple equinox calculations of *Tekufat Shmuel* and *Tekufat Adda*). As the following 4-way comparison chart shows, the vernal equinox according to *Tekufat Adda* was about $3+{}^{1}/_{9}$ days late in *Rambam*'s era, *Tekufat Shmuel* was already about $11+{}^{4}/_{5}$ days late, whereas *Rambam*'s solar longitude = 0° method was by far the most accurate at about ${}^{7}/_{10}$ day early. Even in comparison with his own true solar longitude algorithm *Rambam* ought to have found that in his own era *Tekufat Adda* was about $3+{}^{4}/_{5}$ days late and *Tekufat Shmuel* was about $12+{}^{1}/_{2}$ days late, so it is hard to explain why he so charitably minimized both to only a two-day error! Click here or on the chart to open up a full-page higher-resolution PDF version 43KB (due to limitations of the image generator, the low resolution version has jagged lines for *Tekufat Shmuel* instead of the intended dashed lines):

***Tekufat Shmuel* and *Tekufat Adda* compared**
to the Solar Longitude Methods of *Rambam* and SOLEX 9.1



This chart shows that the lines of *Tekufat Shmuel* and *Tekufat Adda* crossed each other in Hebrew year 2204 (1558-7 BC). **Remarkably, around that same year the seasons according to *Rambam* all crossed the astronomical zero difference line!** In year 2204 *Tekufat Shmuel* and *Tekufat Adda* differed by one minute at *Tekufat Nisan*, by 2 minutes and 2 parts at *Tekufat Tammuz*, by 1 minute and 6 parts at *Tekufat Tishrei*, and by 3 parts = 10 seconds at *Tekufat Tevet*. Nevertheless, their *Tekufat Nisan* and *Tekufat Tevet* were both too early by about 9 days, and their *Tekufat Tishrei* and *Tekufat Tammuz* were both too early by about 12 days. These inaccuracies relative to the actual celestial events would have been even casually obvious, so it should be evident that these *Tekufot* were not actually used as equinox or solstice approximations back then, surely implying that their cross-over was based on back-calculations from later dates, most likely when each method was reasonably accurate.

Compared to the actual numericaly integrated astronomical equinoxes and solstices (referred to mean solar time at the meridian that is halfway between the Nile River and the end of the Euphrates River), *Tekufat Shmuel* was most accurate in year 3408 for spring (353 BC), 3752 for summer (9 BC), 3905 for autumn (144 AD), and 3539 for winter (223 BC), whereas *Tekufat Adda* was most accurate in year 4249 for spring (489 AD), 4726 for summer (966 AD), 5127 for autumn (1366 AD), and 4645 for winter (884 AD). Based on these years, my guesses are that *Tekufat Shmuel* originated in the Second Temple period when it was close to its best accuracy for *Tekufat Tishrei* (autumn), possibly as an initial attempt to regulate the calendar from the beginning of the civil year (*Rosh HaShanah*), and that *Tekufat Adda* originated near the era of Hillel ben Yehudah when it was close to its best accuracy for *Tekufat Nisan* (spring) [by then the inaccuracy of *Tekufat Shmuel* for the spring equinox would have been obvious]. This conjecture suits the debate in *Talmud Bavli* tractate *Sanhedrin* 13a-b, most of which is concerned with basing the leap year decision on the timing of

Tekufat Tishrei with respect to *Sukkot* and *Hoshana Rabbah*, but the abrupt bottom-line ruling at the end of that debate was for *Tekufat Nisan* — no reason was given, other than "don't worry" [about any other considerations]. *Amorah Shmuel* lived in the century before Hillel ben Yehudah, suggesting that he did not originate the *Tekufat Shmuel* calculation that is traditionally attributed to him. *Rav Adda bar Ahavah* lived in the same era as Hillel ben Yehudah.

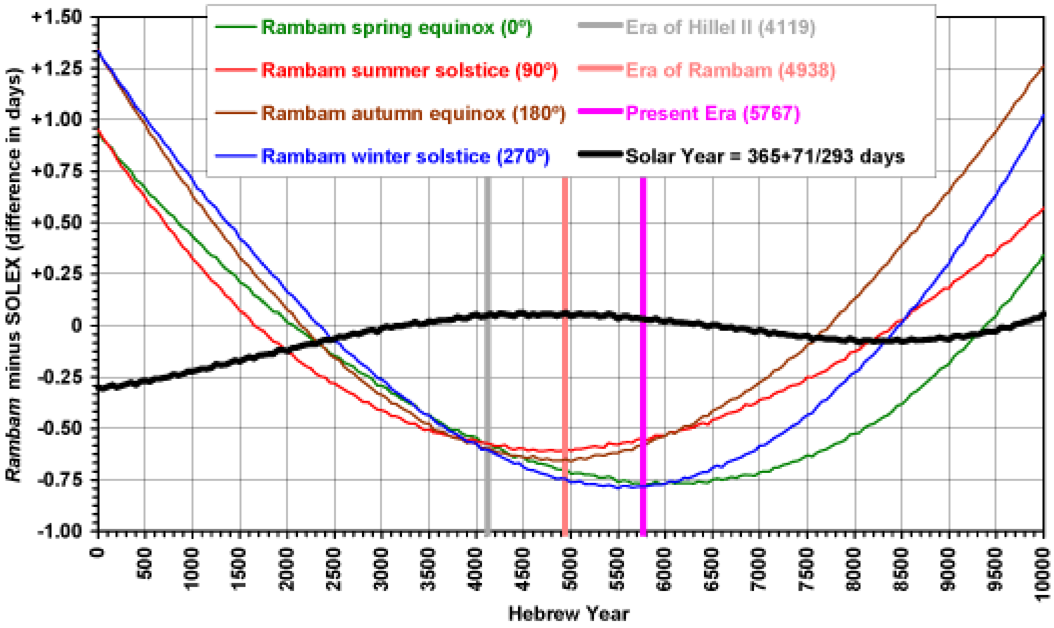
Why were *Tekufat Shmuel* and *Tekufat Adda* made to agree in Hebrew year 2204? What was special about that year? According to *Seder Olam Rabbah*, Yaakov and his family came back from Padan-aram in the year 2205 (Rachel died enroute after giving birth to Benjamin), and that was probably considered to be the year that Yaakov resumed being responsible for intercalation of the calendar, according to *Pirkei D'Rebbi Eliezer*, because if there was a Jewish person in the Land of Israel then Yaakov couldn't have been responsible while outside of Israel.

For the traditional date of Creation, *Rambam*'s true solar longitude method for the vernal equinox was less than one day late, profoundly more accurate than either *Tekufat Shmuel* (about 26 days too early) or *Tekufat Adda* (about $19+{}^{1}/_{2}$ days too early). It is strange that *Rambam* didn't comment about those differences, perhaps he didn't believe his own results!

The *beta* version of my freeware Windows program, *Kalendis*, has the built-in ability to generate a *Tekufot* report for any Hebrew year. Click here to see an example of such a report for Hebrew year 5769, showing astronomical clock times calculated for Israel civil time, along with the traditional *Tekufot* moments reckoned according to the methods of *Amorah Shmuel*, *Rav Adda bar Ahavah*, and *Rambam.*

Present Era Fixed Arithmetic Approximations

In the present era, a fixed arithmetic calendar leap cycle having a mean year of about 365 days 5 hours and 49 minutes can maintain good long-term alignment with the Northward Equinox, because the Northward Equinoctial year length will be almost constant for the next 2 to 3 millennia. A slightly shorter mean year of $365+{}^{71}/_{293}$ days is an excellent approximation for the mean Northward Equinoctial year throughout the first 10000 years of the traditional Hebrew calendar, as shown in the chart below (starting from 195 days 22 hours and 1 minute after the traditional Hebrew calendar epoch) in comparison with *Rambam*'s true solar longitude method:



On the other hand, no simple fixed arithmetic leap cycle can stay aligned with the Southward Equinox, because the Southward Equinoctial year length is changing too rapidly and will continue to do so for about another 5 millennia. This is very well depicted in the charts labelled as #6 of my web page "**The Lengths of the Seasons**" at <http://www.sym454.org/seasons/>. To accurately calculate the moment of the Southward Equinox (solar longitude 180°) one must employ either a mean astronomical polynomial or a more complex astronomical algorithm in terms of Terrestrial Time, then then subtract an appropriate approximation of Delta T to account for the tidal slowing of Earth's rotation rate. Alternatively, a Linear Approximation Progressive Leap Rule, such as LASEY (**L**inear **A**pproximation of the **S**outhward **E**quinoctial **Y**ear) can closely approximate the Southward Equinoctial year using fixed arithmetic and a "smidgen" of calculus, as explained on the "Solar Calendar Leap Rule Studies" at <http://www.sym454.org/leap/>.

This page updated 16 *Cheshvan* 5772 (Traditional) = 16 *Cheshvan* 5772 (Rectified) = Nov 13, 2011 (Symmetry454) = Nov 11, 2011 (Symmetry010) = Nov 12, 2011 (Gregorian)